Self-consistent perturbation theory for random matrix ensembles

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Abstract. An arbitrary but known random matrix ensemble is subjected to a perturbation by any of the three classical random matrix ensembles (GOE, GUE and GSE). Using a BBGKY hierarchy for the correlation functions of the eigenvalues, we propose a self-consistent perturbation expansion and give the result for the two-point function to lowest order in integral form. By way of illustration, the integral is solved for the special case of a Poisson ensemble perturbed by any of the classical ensembles, thereby recovering a result previously derived by other methods.

1. Introduction

In recent years, random matrix theory has found a wide range of applications including nuclear, atomic, molecular and solid state physics, as well as quantum mechanical aspects of chaos. The classical ensembles [1], characterised in the framework of the theory of many complex variables by Cartan, were introduced in physics by Dyson and Mehta [2]. They are known as the Gaussian orthogonal, unitary and symplectic ensembles. Their spectral distributions are well understood. In addition to these, matrix ensembles with random eigenvalues, known as Poisson ensembles, are often used.

With time, an increasing need has arisen to study transitions from one of these ensembles to another, as well as perturbations of arbitrary spectra by these ensembles, as systems in transitional states became available in experiments or through numerical analysis. Several methods are available for such studies; among them the Grassmann variable techniques are very powerful, and results concerning these questions are becoming available [3, 4]. Yet these techniques are cumbersome and one is tempted to use perturbation theory, at least for small deviations. Unfortunately, the two-point function obtained by standard perturbation theory is singular at small distances, because for near-degenerate levels, any perturbation has a strong effect. By mapping the problem on a dynamical model similar to the Brownian motion model used by Dyson [5], we were able to use a self-consistency argument and obtain a two-point correlation function correct for all distances.

In section 2, we develop the model. In section 3, we derive an exact relationship between the correlation functions, well known in statistical mechanics as the BBGKY hierarchy [6]. We develop an approximation scheme for this hierarchy which holds for small times, i.e. in the original problem, for small perturbations. This self-consistent approximation is then solved exactly, which leads to a general solution of the problem.

In section 4, we apply the result of the preceding section to the perturbation of a Poisson spectrum by a classical ensemble—the so-called Porter–Rosenzweig model [7]—recovering in this case the results obtained by Tomsovic [8] and French *et al* [9] using a different method. This serves as a test of the method and an illustration of its application.

2. Description of the model

We consider the following problem. Given an ensemble of Hamiltonians with known properties, what are the properties of the ensemble of Hamiltonians obtained by adding a small random perturbation? More precisely, what are the properties of the Hamiltonian:

$$H^{(\lambda)} = H_0 + V^{(\lambda)} \tag{2.1}$$

if H_0 belongs to a known ensemble and the $v_{ij}^{(\lambda)}$ are independent random variables with Gaussian distribution such that:

$$\langle v_{ij}^{(\lambda)} \rangle = 0 \qquad \langle (v_{ij}^{(\lambda)})^2 \rangle = \lambda \quad (i \neq j) \qquad \langle (v_{ii}^{(\lambda)})^2 \rangle = C\lambda \tag{2.2}$$

where $C = 2/\beta$, depending on whether the perturbation $V^{(\lambda)}$ belongs to the orthogonal, unitary or symplectic ensembles (β is 1, 2 or 4 respectively). For a definition of these ensembles and their various properties, see, e.g. [2]. To make the normalisation somewhat more precise, we shall in the following always suppose that the eigenvalues of H_0 have a spacing of the order of one. The number N of these eigenvalues will always be assumed to be large and the value of λ to be small compared to 1, but independent of N. From these assumptions, it immediately follows that the spectrum of H_0 covers a range of the order of N, whereas that of $V^{(\lambda)}$ covers a range of the order of $\lambda\sqrt{N}$, using standard results on the eigenvalue distribution of GOE matrices.

Because of the statistical properties of the $V^{(\lambda)}$, it is possible to build up $H^{(\lambda)}$ by successive addition of infinitesimal perturbations. This is a limiting form of the following decomposition:

$$H^{(\lambda)} = H_0 + \sum_{k=1}^{n} V_k^{(\lambda/n)}. \tag{2.3}$$

Thus we can use second-order perturbation theory at each step, eventually deriving a differential equation in λ for the joint probability distribution function of the eigenvalues. More precisely, we proceed as follows. Note first that

$$H^{(\lambda+\Delta\lambda)} = H^{(\lambda)} + V^{(\Delta\lambda)} \tag{2.4}$$

where $V^{(\Delta\lambda)}$ has the same properties as described above for the $v_{ij}^{(\lambda)}$ if they are evaluated with respect to the basis of the eigenvectors of $H^{(\lambda)}$. This follows from the fact that these properties are, in fact, invariant under an arbitrary orthogonal (or unitary or symplectic respectively) transformation. Note that this requires that the perturbing ensemble be invariant under those transformations that diagonalise the unperturbed ensemble. Thus, our approach is not applicable, say, to the case of a GUE perturbed by a GOE.

Under these circumstances, one can write down an equation for the eigenvalues $x_i(\lambda)$

$$x_i(\lambda + \Delta\lambda) = x_i(\lambda) + v_{ii}^{(\Delta\lambda)} + \sum_{j=1}^{N}{}' \frac{|v_{ij}^{(\Delta\lambda)}|^2}{x_i(\lambda) - x_j(\lambda)}. \tag{2.5}$$

Since the perturbation is random Gaussian, the above equation is seen to be quite similar to a Langevin equation, so that we can follow the familiar steps that lead to a Fokker–Planck equation for the probability distribution of the eigenvalues. Denoting the joint probability distribution function for the $x_i(\lambda)$ by $P_\lambda(x_i)$, one finds

$$\frac{\partial P}{\partial \lambda} = -\text{div}\, j \tag{2.6}$$

where

$$(j)_i = -\frac{C}{2}\frac{\partial P}{\partial x_i} + \sum_{j=1}^{N}{}' \frac{1}{x_i - x_j} P. \tag{2.7}$$

The derivation of this equation is straightforward and follows standard methods. For completeness' sake, however, it is presented in the appendix. Note that we have dropped the subscript λ, as we shall continue to do in the following.

The physical interpretation of these equations is now straightforward: the joint probability distribution function of the eigenvalues satisfies a continuity equation, where the particle current j consists of two parts: the first one is an ordinary diffusive current, with a diffusion constant $C/2$, and the second part is a drift contribution where the velocity of the motion is determined by the sum $\sum_{j=1}^{N}{}' (x_i - x_j)^{-1}$. The motion of the eigenvalues can therefore be described as random motion of particles interacting with a force F given by the following expression:

$$F = -\nabla W \qquad W = -\frac{1}{2}\sum_{i,j=1}^{N}{}' \ln|x_i - x_j| \tag{2.8}$$

where we have assumed that the particles (eigenvalues) are moving in a very viscous medium, so that inertial effects may be neglected and the velocity is indeed proportional to the force. This is obviously very similar to the Brownian motion model introduced by Dyson (see, e.g., [5] for greater details). In fact, the only difference is the absence, in our model, of a harmonic restoring force; this may be traced directly back to an issue of normalisation which is of no particular interest to us. The use to which we shall put it, however, is slightly different; we shall evaluate the short-time response of a system starting very far from equilibrium. The simplifying factor in this case is the fact that we know the initial conditions and can assume that they will not be violently modified in the first few instants. In the following, we derive exact equations for the correlation functions (in essence, the BBGKY hierarchy for this system) as well as a generally workable approximation for short-time behaviour, which we later shall apply to an ensemble of matrices with randomly distributed eigenvalues (Poisson ensemble) perturbed by a GOE.

3. The correlation functions

The correlation functions for the eigenvalues are defined as follows:

$$\rho_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int dx_{n+1} \ldots dx_N P(x_1, \ldots, x_N). \tag{3.1}$$

The usefulness of knowing the correlation functions rather than the full distribution

arises from the fact that frequently the quantities of interest do not involve more than a few particles at a time. In fact, the statistics most commonly used in characterising matrix ensembles can (with one exception) all be obtained from the knowledge of the correlation functions with $n \leq 4$. The exception is the nearest-neighbour spacing distribution, which requires a knowledge of *all* correlation functions. This must, therefore, usually be considered separately.

To derive an equation for the correlation functions, it is sufficient to integrate the equation for $P(x_1, \ldots, x_N)$ over the appropriate variables. Splitting all the occurring sums in two parts (one running over the external variables and another over the integrated ones), one readily obtains

$$\frac{d}{d\lambda} \rho_n(x_1, \ldots, x_n)$$

$$= \frac{C}{2} \sum_{i=1}^{n} \frac{\partial^2 \rho_n}{\partial x_i^2} - \sum_{i,j=1}^{n}{}' \frac{\partial}{\partial x_i} \left(\frac{1}{x_i - x_j} \rho_n \right)$$

$$- \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\int_{-\infty}^{\infty} dx_{n+1} \frac{1}{x_i - x_{n+1}} \rho_{n+1}(x_1, \ldots, x_{n+1}) \right) \tag{3.2}$$

where the integral term is to be understood as a principal value, reflecting the fact that the corresponding sums were always taken excluding the singular term. Thus we see that the equation for the n-particle correlation function requires the knowledge of the $(n+1)$-particle correlation function. The physical reason for this is the following: the presence of the particles at the fixed coordinate $x_1, \ldots, x_n$ implies that these particles will interact with each other by the direct effect of the interaction potential as well as indirectly, through the effect of all other particles rearranging in response to the presence of the n particles. This last effect is the one described by the integral term in (3.2), and for its description the knowledge of higher-order correlations is clearly needed. Thus one needs some way of closing the equations. This is usually done by an ansatz typically involving factorisation assumptions on higher-order correlation functions, expressing them in terms of lower-order ones. Unfortunately, the exact range of validity of such approximations cannot be determined *a priori.*

In our case, however, we are only interested in the small-time behaviour of the system (i.e. we want the behaviour for small strength of the perturbation parameter λ). It is therefore possible to make a systematic perturbation expansion in λ. In the entire paper, we shall limit ourselves to first order, but it should be borne in mind that, using a larger number of correlation functions, it is quite possible to carry the expansion further. Let us determine the two-particle correlation function $\rho_2(x_1, x_2)$ to first order. The simplest approximation would be, naturally, to insert the known correlation functions $\rho_2^{(0)}(x_1, x_2)$ and $\rho_3^{(0)}(x_1, x_2, x_3)$ in the right-hand side of (3.2). This approximation may, however, be inadequate under certain circumstances. The reason is that the interaction potential W is singular at small interparticle distances. The consequence is that, even in a very short time, eigenvalues originally very close to one another may be very strongly affected. Under such circumstances, however, it appears reasonable to assume that the crucial influence is the one of the direct interaction between the two particles, as opposed to the interaction between the two particles and the cloud of correlated particles surrounding them. One therefore obtains a good description of the situation by approximating the integral term by its (supposedly known) unperturbed value while keeping the direct interactions exactly. This yields

the following short-time approximation to (3.2):

$$\frac{d}{d\lambda} \rho_n(x_1, \ldots, x_n)$$

$$= \frac{C}{2} \sum_{i=1}^{n} \frac{\partial^2 \rho_n}{\partial x_i^2} - \sum_{i,j=1}^{n}{}' \frac{\partial}{\partial x_i} \left(\frac{1}{x_i - x_j} \rho_n \right)$$

$$- \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\int_{-\infty}^{\infty} dx_{n+1} \frac{1}{x_i - x_{n+1}} \rho_{n+1}^{(0)}(x_1, \ldots, x_{n+1}) \right). \tag{3.3}$$

Let us now consider only the functions $\rho_1(x_1)$ and of $\rho_2(x_1, x_2)$. In principle, we could separate density-dependent terms and use other types of n-point functions that display translational invariance upon a variable density background [2], but for compactness we shall restrict our considerations to an energy interval sufficiently narrow that density variations within it may be ignored. We shall use relative coordinates such as $x = x_2 - x_1$ and $y = x_3 - x_1$. The function ρ_1 is then independent of position, ρ_2 depends exclusively on x and ρ_3 on x and y. For the density we therefore find

$$\frac{d\rho_1}{d\lambda} = \frac{\partial}{\partial x_1} \int_{-\infty}^{\infty} \frac{\rho_2(x_1, x_2)}{x_2 - x_1} dx_2$$

$$= \frac{\partial}{\partial x_1} \int_{-\infty}^{\infty} \frac{\rho_2(x)}{x} dx$$

$$= 0 \tag{3.4}$$

since $\rho_2(x)$ is an even function of x and the integral is taken as a principal value. Thus, the local level density remains unaffected by the perturbation. (This remark does not hold, of course, for the variations of the density which may occur within a number of spacings proportional to N.)

Similarly, we find for the two-point function

$$\frac{\partial \rho_2}{\partial \lambda}(x; \lambda) = C \frac{\partial^2 \rho_2}{\partial x^2} - 2 \frac{\partial}{\partial x} \left(\frac{\rho_2(x)}{x} \right) + G(x)$$

$$G(x) = -\frac{\partial}{\partial x} \left(\int dy \frac{\rho_3^{(0)}(x, y)}{y} + \int dy \frac{\rho_3^{(0)}(x, y)}{x - y} \right). \tag{3.5}$$

This is a linear equation which is readily solved as follows. Define the functions $\phi_k(x)$ as follows: they are the normalised solutions to the eigenvalue problem

$$C \frac{d^2\phi}{dx^2} - 2 \frac{d}{dx} \left(\frac{\phi_k}{x} \right) = -k^2 \phi_k(x) \tag{3.6}$$

which has an unnormalised solution for every real k. The operator is self-adjoint with respect to the scalar product

$$(f, g) = \int_0^{\infty} x^{-2/C} f(x) g(x) \, dx \tag{3.7}$$

so that the system $\phi_k(x)$ is a complete orthonormal system under this product. Therefore every function can be expressed as a linear combination of ϕ_k. We define $a(k, \lambda)$ and $b(k)$ as follows:

$$\rho_2(x; \lambda) = \int a(k, \lambda) \phi_k(x) \, dk \qquad G(x) = \int b(k) \phi_k(x) \, dk. \tag{3.8}$$

The solution of the original system is now reduced to finding $a(k, \lambda)$ knowing both $a(k, 0)$ and $b(k)$. The equation for $a(k, \lambda)$ is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} a(k, \lambda) = -k^2 a(k) + b(k) \tag{3.9}$$

with the solution

$$a(k, \lambda) = (a(k, 0) - b(k)/k^2) \exp(-k^2\lambda) + b(k)/k^2. \tag{3.10}$$

This solves the problem completely by quadratures, since the $\phi_k(x)$ are known in terms of elementary functions. The equations defining $a(k, 0)$ and $b(k)$ can be inverted using the assumed orthonormality of the $\phi_k(x)$. Before we proceed to the more special applications, however, let us look at certain quite general consequences of these formulae.

The functions $\phi_k(x)$ are explicitly given by the following general formula:

$$\phi_k(x) = k^{-1/C} C^{-(1-2/C)/4} \Phi(kx/\sqrt{C})$$
$$\Phi(x) = x^{(1+2/C)/2} J_\nu(x) \qquad \nu = \tfrac{1}{2}|1 - 2/C|. \tag{3.11}$$

We can express the general form of the solution in terms of these Bessel functions and of the amplitudes $a(k, 0)$ and $b(k)$

$$\rho_2(x) = \int a(k, \lambda) \phi_k(x) \, \mathrm{d}k$$

$$= C^{-(1-2/C)/4} \int k^{-1/C} [(a(k, 0) - b(k)/k^2) \exp(-k^2\lambda) + b(k)/k^2]$$

$$\times \Phi(kx/\sqrt{C}) \, \mathrm{d}k. \tag{3.12}$$

For small x, the following can be seen quite generally by expanding the Bessel functions in (3.11): for $C \leq 2$ the function $\Phi(x)$ goes as $x^{2/C}$ as $x \to 0$. From this and the above equation follows immediately

$$\rho_2(x) \sim \text{constant} \times x^{2/C} \tag{3.13}$$

where the constant is in general non-zero. From this it follows that, in the most general case, an ensemble perturbed by a GOE, GUE or GSE shows at small distances the type of eigenvalue repulsion exhibited by the perturbing ensemble. This is hardly surprising if the original ensemble had no (or less strong) eigenvalue repulsion. It is more surprising in the opposite case, where the original ensemble has a stronger repulsion between levels than the perturbing one. In this case, one has from the definition of $a(k, 0)$ that

$$\int_0^\infty k^{1/C} a(k, 0) \, \mathrm{d}k = 0 \tag{3.14}$$

so that the initial $\rho_2^{(0)}(x)$ has this type of level repulsion. In general, however, (3.14) does not appear to remain true for non-zero λ and the constant defined above is generally non-zero. This has the remarkable consequence that an ensemble with, say, $\rho_2^{(0)}(x) \sim x^2$ as $x \to 0$, once perturbed by a GOE, has more eigenvalues close to each other, i.e. $\rho_2(x) \sim \text{constant} \times x$ where the constant is quite small in λ, however. This

is presumably due to the presence in the perturbing matrix, of a finite concentration of elements of size comparable to the spacing of the eigenvalues in the unperturbed system. These will be able to shift the eigenvalues by amounts comparable to their separation, thereby overcoming the effect of the original strong repulsion, replacing it instead by a weaker repulsion with a very small prefactor. As these matrix elements are only present with an exponentially small probability, the effect is not likely to be observable in practice, however. Note, though, that the perturbed ensemble must be diagonalisable by the symmetry operations of the perturbing ensemble; thus, for example, the case of a GUE perturbed by a GOE cannot be handled by our methods and the above remarks do not apply. On the other hand, they are expected to apply to the case of a picket fence perturbed by a GOE, if appropriately taken as a limiting case.

4. The Porter–Rosenzweig model

We now turn our attention to the Porter-Rosenzweig model, i.e. the case of the Poisson ensemble perturbed by a small GOE. The Poisson ensemble is defined as an ensemble of matrices with no correlation between the eigenvalues. One therefore has

$$\rho_n(x_1, \ldots, x_n) = \rho^n \tag{4.1}$$

where ρ is simply the eigenvalue density, which we take to be a constant. Note that we do not have any delta functions, because the correlation functions, as we have defined them in the previous section, refer to the probability of finding n *different* eigenvalues in the positions $x_1, \ldots, x_n$. From this it follows that the three-body term in (3.3) is zero, and therefore so are the coefficients $b(k)$ occurring in (3.10). This is because $\rho^3 \int \mathrm{d}y/y = 0$ if the integral is taken to be a principal value, as we have assumed throughout.

The only remaining task is therefore to compute the $a(k, 0)$. This leads to integrals which can be evaluated straightforwardly to give the following expressions for $\rho_2(x; \lambda)$, where, for convenience, we have set ρ equal to 1:

$$\rho_2(x; \lambda) = \begin{cases} \dfrac{x}{\sqrt{2}} \displaystyle\int_0^\infty \exp(-k^2\lambda) J_0\left(\dfrac{kx}{\sqrt{2}}\right) \mathrm{d}k & C = 2 \\[18pt] x \displaystyle\int_0^\infty \sin kx \exp(-k^2\lambda) \, \mathrm{d}k & C = 1 \\[18pt] \sqrt{2}x \displaystyle\int_0^\infty (1 - \lambda k^2) \exp(-k^2\lambda) \sin(\sqrt{2}kx) \, \mathrm{d}k & C = \tfrac{1}{2} \end{cases} \tag{4.2}$$

thereby recovering the results of [8]. Note that these functions can all be cast into the dimensionless form

$$\rho_2(x; \lambda) = F\left(\frac{x}{\sqrt{\lambda}}\right) \tag{4.3}$$

which means that the only relevant energy scale in the form of the correlation functions of the Porter-Rosenzweig model is the typical size of the matrix elements of the perturbation. Since there were no correlations to start with, this is perhaps not very surprising. The functions are shown in figure 1. The short-range behaviour is, as
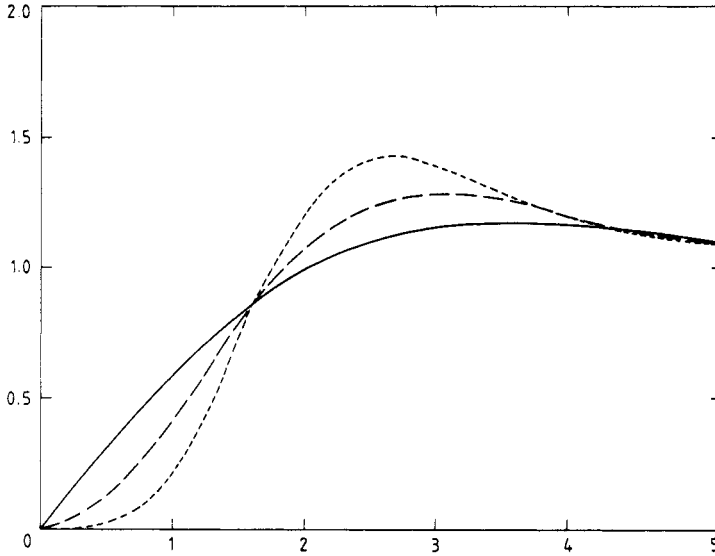
Figure 1. Plot of the two-point correlation function of a Poisson ensemble perturbed by a GOE (full curve), a GUE (broken curve) and a GSE (dotted curve) as a function of the dimensionless distance between eigenvalues $x/\sqrt{\lambda}$.

expected, very different for the three models, but the long-range behaviour appears very similar. This is easily confirmed by an explicit computation which yields

$$\rho_2(x; \lambda) = 1 + \frac{2\lambda}{x^2} + O(x^{-3}).\tag{4.4}$$

The form of the function can also be understood on physical grounds: the repulsion between the particles is responsible for the depletion of the particles close to the origin. However, since the original spectrum did contain eigenvalues quite close to the origin and since the 'time' to move was quite short, the particles accumulated at a distance of order $\sqrt{\lambda}$, which corresponds to the distance a diffusing particle covers in a time λ.

As a final remark, the following point should be noted: it is, of course, possible to compute such characteristics as the number variance $\Sigma_2(L)$ (see, e.g., [2] for a definition) from the two-point correlation function. In this case, it can be shown that

$$\lim_{L\to\infty} \frac{\Sigma_2(L)}{L} = 1 + \int_{-\infty}^{\infty} (\rho_2(x; \lambda) - 1)\, dx.\tag{4.5}$$

From the equations we have derived, it is found that this last integral is always independent of λ for the reason that the equation determining $\rho_2(x; \lambda)$ is of the form

$$\frac{d\rho_2}{d\lambda} = \frac{d}{dx} F(\rho_2, x).\tag{4.6}$$

This holds for the original equation (which is an exact relation) as well as for all its later approximations. Therefore, it follows that the asymptotic behaviour of the number variance as defined above can never be affected by a small perturbation. Thus the effects of small perturbations are strictly limited to correspondingly small energy intervals. This result is in agreement with numerical results of Roman [10].

5. Conclusions

We have mapped the problem of an aribtrary known ensemble of Hamiltonians perturbed by a GOE (or GUE or GSE) onto a dynamical problem of interacting particles, which we were able to solve for small times, i.e. small perturbations. By using the BBGKY hierarchy and solving self-consistently for the two-point function, we were able to obtain a result valid for all distances, as opposed to standard perturbation theory, which only recovers the long-distance behaviour correctly.

The solution is given in integral form, and its short-distance behaviour can be given generically. It depends on the perturbation only, and is linear for the GOE, quadratic for the GUE and quartic for the GSE. This behaviour translates directly into a corresponding property of the nearest-neighbour distribution. This is due to the fact that two eigenvalues that are very close one to the other are overwhelmingly likely to be nearest neighbours, so that the short-distance behaviour of the nearest-neighbour distribution is the same as that of the two-point function. This short-range behaviour for the nearest-neighbour distribution function is not unexpected, yet functional forms with other types of short-range behaviour have been occasionally used in the literature. Furthermore, using the general result, special cases of interest may be treated; by way of example, we give the result for a Poisson distribution perturbed by a GOE, GUE or GSE. This exactly confirms the result previously obtained by other means by Tomsovic [8] and French *et al* [9].

Obviously, the general integral solution can be applied to many other models. Thus, following an idea by Berry and Robnik [11], the properties of partially chaotic systems would be determined by a direct sum of GOE and a Poisson ensemble. The short-range behaviour of this model is not satisfactory [12] and we propose to improve it by introducing a GOE perturbation of the type discussed in this paper and compare the integrals (3.13) with numerical experiments. Similarly we can discuss weakly coupled spectra (due to slight symmetry breaking) such as may occur in nuclear or molecular physics. Direct sums of GOE of different relevant sizes will be needed, and again the solution is readily available in terms of our general results.

Appendix

In order to obtain the evolution of $P_\lambda(x_1, \ldots, x_N)$ as a function of λ, one starts from the transformation law for a probability density, namely that

$$P_\lambda(x_1(\lambda), \ldots, x_N(\lambda)) \, dx_1(\lambda) \ldots dx_N(\lambda) \tag{A1}$$

is independent of λ. From this it follows that

$$P_{\lambda+\Delta\lambda}(x_1(\lambda + \Delta\lambda), \ldots, x_N(\lambda + \Delta\lambda))$$
$$= J(x_1(\lambda), \ldots, x_N(\lambda)) P_\lambda(x_1(\lambda), \ldots, x_N(\lambda)) \tag{A2}$$

where

$$J(x_1(\lambda), \ldots, x_N(\lambda)) = \frac{\partial(x_1(\lambda), \ldots, x_N(\lambda))}{\partial(x_1(\lambda + \Delta\lambda), \ldots, x_N(\lambda + \Delta\lambda))}. \tag{A3}$$

To simplify the notation, we shall henceforth denote $x_i(\lambda)$ by x_i and $x_i(\lambda + \Delta\lambda)$ by y_i, dropping the λ dependence.

We now wish to change (A2) so that it becomes a relation between $P_{\lambda+\Delta\lambda}$ and P_λ, both evaluated at the points $y_1, \ldots, y_N$. For this purpose, we need to evaluate the Jacobian as a function of the y_i as well as the function $P_\lambda(x_1, \ldots, x_N)$.

To this end we need the expression of second-order perturbation theory, i.e. equation (2.5), in order to obtain the y_i as a function of the x_i, which in first order can be inverted as follows

$$x_i = y_i - v_{ii}^{(\Delta\lambda)} - \sum_{j=1}^{N} {}' \frac{|v_{ij}^{(\Delta\lambda)}|^2}{y_i - y_j}. \tag{A4}$$

We first evaluate the Jacobian of this transformation for an arbitrary perturbation $V^{(\Delta\lambda)}$, not averaging until the end. Roughly speaking, then, $v_{ij}^{(\Delta\lambda)}$ is of the order of $(\Delta\lambda)^{1/2}$, so that the above Jacobian must be evaluated to second order in the perturbation. This gives

$$J(x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{\partial x_i}{\partial y_i}$$

$$= \prod_{i=1}^{N} \left(1 - \frac{\partial}{\partial y_i} \sum_{j=1}^{N} {}' \frac{|v_{ij}^{(\Delta\lambda)}|^2}{y_i - y_j} \right)$$

$$= 1 - \sum_{i,j=1}^{N} {}' \frac{\partial}{\partial y_i} \frac{|v_{ij}^{(\Delta\lambda)}|^2}{y_i - y_j} + O((\Delta\lambda)^2). \tag{A5}$$

We now need to develop $P_\lambda(x_1, \ldots, x_N)$ in a Taylor series to second order in the $v_{ij}^{(\lambda)}$. This yields

$$P_\lambda(x_1, \ldots, x_N) = P_\lambda(y_1, \ldots, y_N) - \sum_{i=1}^{N} v_{ii}^{(\lambda)} + \frac{1}{2} \sum_{i,j=1}^{N} v_{ii}^{(\lambda)} v_{jj}^{(\lambda)} \frac{\partial^2 P_\lambda}{\partial y_i \, \partial y_j}$$

$$- \sum_{i,j=1}^{N} {}' \frac{|v_{ij}^{(\Delta\lambda)}|^2}{y_i - y_j} \frac{\partial P_\lambda}{\partial y_i} \tag{A6}$$

again up to terms of higher order in $\Delta\lambda$.

We now average (A5) and (A6) over the possible values of the $v_{ij}^{(\lambda)}$, obtaining thereby

$$J(x_1, \ldots, x_N) = 1 - \Delta\lambda \sum_{i,j=1}^{N} {}' \frac{\partial}{\partial y_i} \frac{1}{y_i - y_j} \tag{A7}$$

and for $P_\lambda(x_1, \ldots, x_N)$:

$$P_\lambda(x_1, \ldots, x_N) = P_\lambda(y_1, \ldots, y_N) + \frac{C}{2} \Delta\lambda \sum_{i=1}^{N} \frac{\partial^2 P_\lambda}{\partial y_i^2} - \Delta\lambda \sum_{i,j=1}^{N} {}' \frac{1}{y_i - y_j} \frac{\partial P_\lambda}{\partial y_i} \tag{A8}$$

the equations (A2), (A7) and (A8) are then readily combined to yield the result mentioned in the text. Note, however, that while the original problem was invariant under a change from λ to $-\lambda$, the final Fokker-Planck equation clearly is only valid for positive λ. This comes from the fact that we have systematically expressed everything in terms of the y_i. Had we done the opposite and expressed everything in terms of the x_i, we would have obtained the time-reversed equation.

References

[1] Hua L K 1963 *Harmonic Analysis of Functions of Several Complex Variables* (Translated by L Ebner and A Korányi) (Providence, RI: American Mathematical Society)
[2] Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic)
[3] Verbaarschot J J M, Weidenmüller H A and Zirnbauer M 1985 *Phys. Rep.* **129** 367
[4] Nishioka H, Verbaarschot J J M, Weidenmüller H A and Yoshida S 1987 *Ann. Phys., NY* **172** 67
[5] Dyson F J 1962 *J. Math. Phys.* **3** 1191
[6] Hill T H 1956 *Statistical Mechanics* (New York: McGraw-Hill)
[7] Rosenzweig N and Porter C E 1960 *Phys. Rev.* **120** 1698
[8] Tomsovic S 1986 *Thesis* University of Rochester
[9] French J B, Kota V K B, Pandey A and Tomsovic S 1988 *Ann. Phys., NY* **181** 188
[10] Roman C 1986 *Thesis* Universidad Nacional Autónoma de México
[11] Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
[12] Seligman T H and Verbaarschot J J M 1985 *J. Phys. A: Math. Gen.* **18** 2227